Symmetric periodic motions in a ring problem

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1 Introduction

The motion in a multi-body system is generally very complicated. However, we can obtain valuable information on similar physical systems, by using simpler N-body models based on geometrical properties of the system [2]. In the present work, we study the motion of a small body under the gravitational attraction of an 8-body ring configuration. Among the existing periodic orbits, those which are symmetric with respect to the x-axis of the synodic coordinate system are of special interest. As far as their stability is concerned, their evolution has been studied for $\beta = 2$ by varying the Jacobian constant. We have found stable as well as unstable orbits, which will be studied thoroughly later by varying the mass ratio β.

2 Equations of motion

The general form of an N-body ring configuration consists of a central primary of mass M and ν peripherals of mass m_i $(i = 1, 2, ..., \nu)$, where $m_1 = m_2 = ... = m_\nu$ (thus $N = \nu+1$), which form a regular polygon. For the peripherals' motion around the primary to be circular, the resultant of the gravitational forces must even up the centrifugal force. The common angular velocity of the peripherals is ω and so the regularity of the polygon is preserved. Then, Newton's second law for the motion of a particle, the mass of which is considered negligible (that is, it does not influence the motion of the other bodies), gives the following equations [2]:

$$\ddot{x} - 2\dot{y} = -\frac{\partial U}{\partial x}, \ \ddot{y} + 2\dot{x} = -\frac{\partial U}{\partial x} \tag{1}$$

where U stands for the potential function. There is an evident similarity of the latter with the respective equations of the restricted 3-body problem [5]. The potential function U is spatial-dependent only, and its general expression is:

$$U = \frac{1}{2}(x^2 + y^2) + \frac{1}{\Delta}(\frac{\beta}{r_0} + \sum_{i=1}^{\nu} \frac{1}{r_i}) \tag{2}$$

where β is the mass ratio $\frac{M}{m_i}$ and Δ is a constant of the problem which can easily be estimated by the geometrical properties of the polygon. The r_0, r_i $(i = 1, 2, ..., \nu)$ are the dimensionless distances of the particle from the central primary and the peripherals respectively. The motion described by the set of equations (1) has one integral, given by the following relation:

$$C = 2U - (\dot{x}^2 + \dot{y}^2) \tag{3}$$

where C is the so-called Jacobian constant.

The periodic orbits were determined by numerical integration of the equations (1), using a Runge-Kutta method. Among the existing retrograde periodic orbits, of particular interest are the symmetric ones with respect to the rotating x-axis (with $\dot{x_0} = 0$ and $\dot{y_0} \neq 0$); that is the axis joining the central primary with one peripheral, in the (x-y) plane. The stability of periodic orbits is studied by evaluation of Henon stability indices a, b, c, d, which are functions of the initial values $\dot{x_0}$, $\dot{y_0}$ and C. Then, the condition of symmetry gives a=d, and so the stability condition becomes $\|a\| < 1$. Due to the symmetry of the periodic orbits, the indices are evaluated for the half orbit [3].

3 Numerical results and discussion

The orbits studied in this work are found around the central primary, inside and outside the peripherals' polygon, as well as around one or more peripherals, even twisting and curling among them. For each of the above types of motion, indicative plots in the x-y plane are given, as well as information about their initial conditions and period. Also, their stability is evaluated for mass ratio $\beta = 2$ by varying the Jacobian constant.

3.1 Orbits around the central body and one peripheral

Starting from the periodic orbits surrounding the central body, the family of these orbits is generated from infinitesimal (circular) orbits around the central primary. As the orbits approach the central body and shrink to it, the value of the Jacobian constant goes to infinity. Also, the orbits are generated and evolve around the peripheral 1 in the same way as the ones presented above.

In the two cases, as C decreases, the elliptic-shaped orbits increase more and more in size and draw away from their centre, and after a while a collision orbit with a neighbouring peripheral appears. Concerning their stability, in general the orbits in this region are stable.

3.2 Orbits around all the peripherals

Here, two families of retrograde simple periodic orbits are generated. The generation of these families of periodic orbits is approximately similar to the l and m classes of the restricted 3-body problem correspondingly. The one family of periodic orbits is generated for circle-like orbits far from the peripherals ($r_i >> 1$) and for large positive values of C. The other one is generated also far from the peripherals, but the values of C are negative. As C changes, the periodic orbits shrink to the peripherals and turn to elliptical, up to a value of C where the respective orbit collides with one of them. The orbits of these families calculated in this region in general were also found stable.

All the above cases,as the classes m, l, f, h, g and i of the restricted 3-body problem, present interest for astronomical applications.

3.3 Orbits among the peripherals

The last type of motion studied consists of orbits twisting among the peripherals. Their characteristics vary, depending on their initial conditions. There are orbits surrounding two peripherals, orbits that curve at the central body, orbits curving at one or more peripherals and even orbits localized among a few of them.

In figure 1(a) we see two retrograde periodic orbits curving at peripheral 1, surrounding all the rest bodies, with C=-1.95, x_0=-1.934, T=3.40, a=53.7 for orbit (I), and C=-0.95, x_0=-2.377, T=4.198, a=63.3 for orbit (II). As C decreases, the orbit closes up to all the peripherals except 1, from which it moves away, while the curving smoothens. Figure 1(b) shows two other similar periodic orbits, surrounding only the symmetric peripherals 2 and 7 and the central body. As the Jacobian decreases, the orbits approach more and more the peripherals 2 and 7.

Their characteristic values are: (I) C=-0.100, x_0=-0.275, T=1.7, a=1178.6, and (II) C=2.30, x_0=-0.521, T=1.25, a=641.

The values of the stability index of the periodic orbits of the two families in this region are very large. We should note that orbits 1(a)I, 1(b)I have similar form with some of the classes f and b, of the restricted three body problem [4, 5].

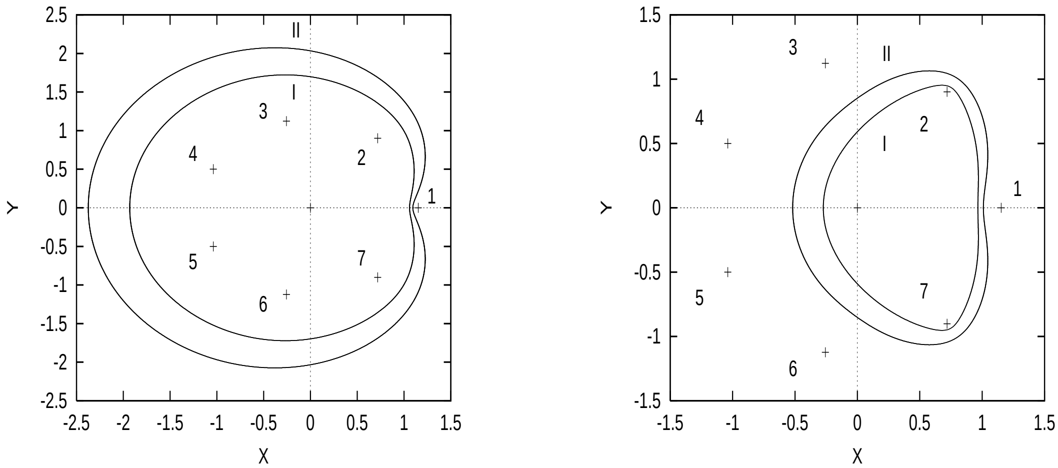

Figure 1

In figure 2(a) we show retrograde periodic orbits with different form from the ones above. These orbits surround all the bodies,except the peripherals 2 and 7. They present a similar behaviour as the previous ones; that is, as C decreases, the two symmetric curvings that exist at peripherals 2 and 7 smoothen and the orbit moves away from them more and more, approaching the peripherals 1, 3, 4, 5 and 6. They have initial conditions C=-0.90, x_0=-1.752 for orbit (I) and C=1.000, x_0=-2.293 for orbit (II). The respective periods and stability coefficients are: (I) T=3.066, a=928.9 and (II) T=4.236, a=1337.37.

Figure 2(b) shows also two similar periodic orbits that do not surround the peripherals 1, 2, 7. These have: (I) C=-2.000, x_0=-1.465 and (II) C=2.050, x_0=-2.491, while the periods and stability coefficients are: (I) T=2.197, a=314.1 and (II) T=4.64, a=381.3. As C decreases, they move more and more near the peripherals 3, 4, 5, 6 and far from 1, 2, 7, surrounding the central body. Also, the two symmetric curvings at 2 and 7 become smooth more and more. The orbits of these two families in this region are very unstable.

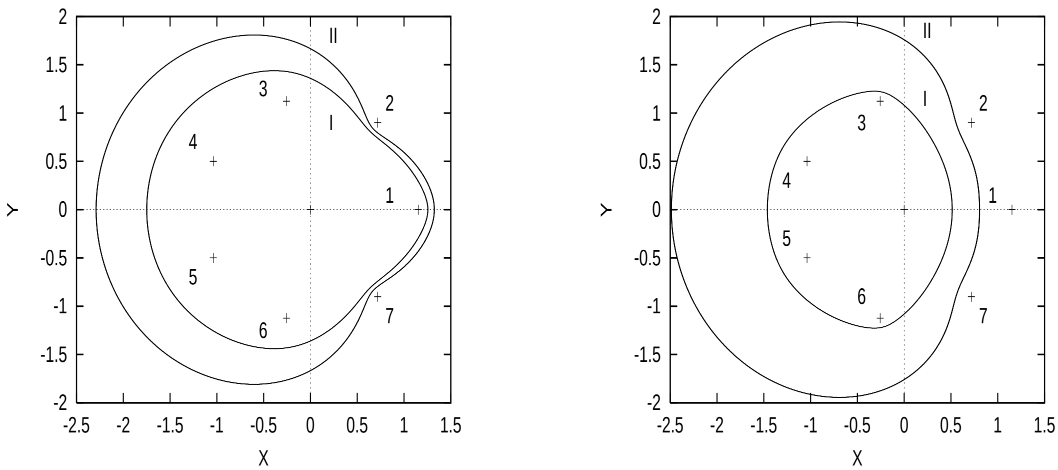

Figure 2

We have found a new family of periodic orbits around all the peripherals except the symmetric ones 3, 6, which is presented in figure 3(a). We have also found another family which surrounds all the peripherals except the symmetric ones 4, 5, shown in figure 3(b). We give the initial conditions, period and stability index for two orbits of each family; for 3(a): (I) C=-2.000, x_0=-1.307, T=2.51, a=3579.8, (II) C=2.100, x_0=-1.773, T=5.88, a=2306.8 and for 3(b): (I) C=-2.000, x_0=-0.820, T=2.74, a=139.7, (II) C=2.500, x_0=-1.004, T=5.71, a=626.1. These two families are very unstable in these regions. As C changes, their orbits move more and more to the internal or external peripherals, as the Jacobian constant decreases or increases correspondingly.

Further development of our calculations is associated with the unstable character of two new families of retrograde periodic orbits. In figure 4(a) we give two orbits around the central body and the symmetric peripherals 4, 5, with: (I) C=0.500, x_0 =-1.271, T=1.404, a=180.9, and (II) C=2.000, x_0=-1.458, T=2.11, a=30.7. Also figure 4(b) shows two orbits surrounding the peripherals 1, 2, 7, as well as the central body, with: (I) C=-2.000, x_0=-0.195, T=1.6, a=809.9, and (II) C=2.500, x_0=-0.648, T=4.02, a=262.8. In the two cases, as C decreases the orbits approach more and more the internal peripherals, while moving to the external ones as C increases.

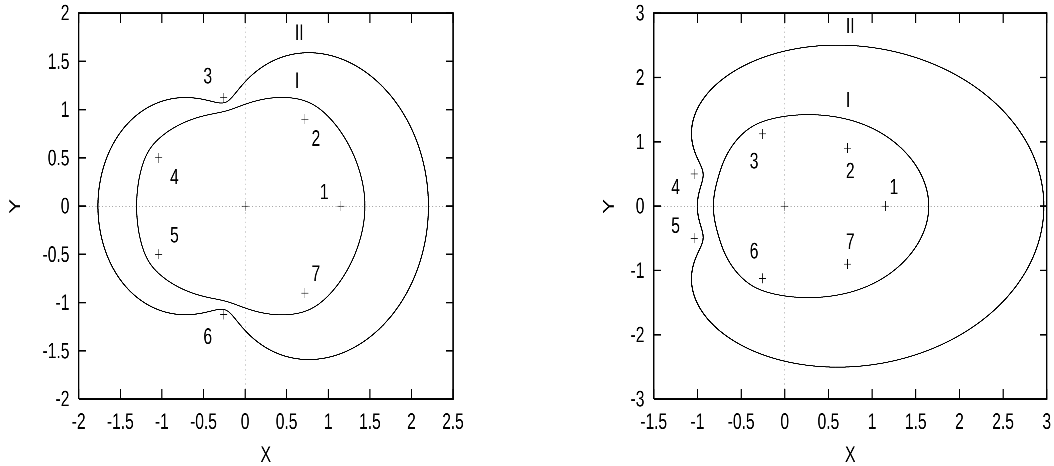

Figure 3

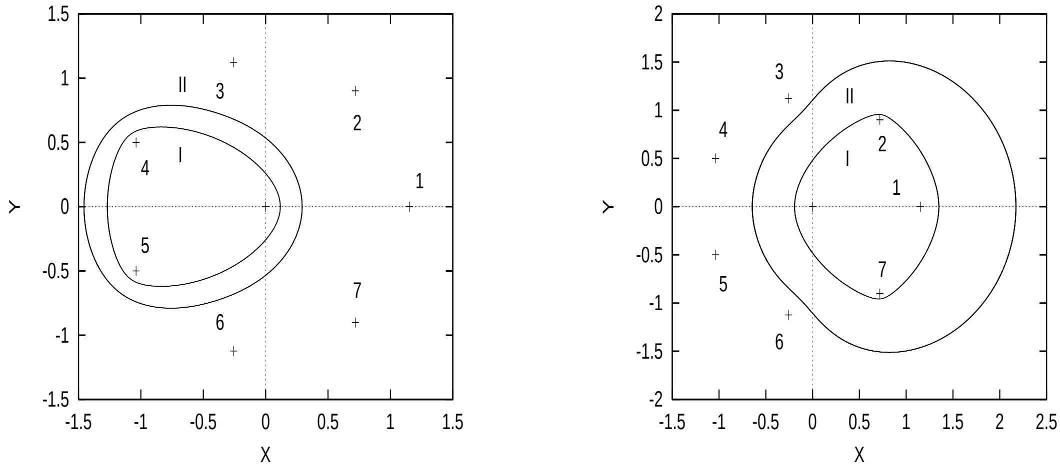

Figure 4

Regarding the retrograde motion around two peripherals, figure 5(a) shows two orbits with (I) C=-1.700, x_0=-1.141, T=0.61, a=443.2 and (II) C=1.700, x_0=-1.194, T=0.78, a=171.8, while figure 5(b) shows two orbits which enclose also the central body, with: (I) C=-1.450, x_0=-0.746, T=1.76, a=2235, and (II) C=2.500, x_0=-0.863, T=2.76, a=1808. As C decreases, the orbits draw near more and more to the two peripherals, and for certain values of C we have a collision with them. All these orbits are also unstable.

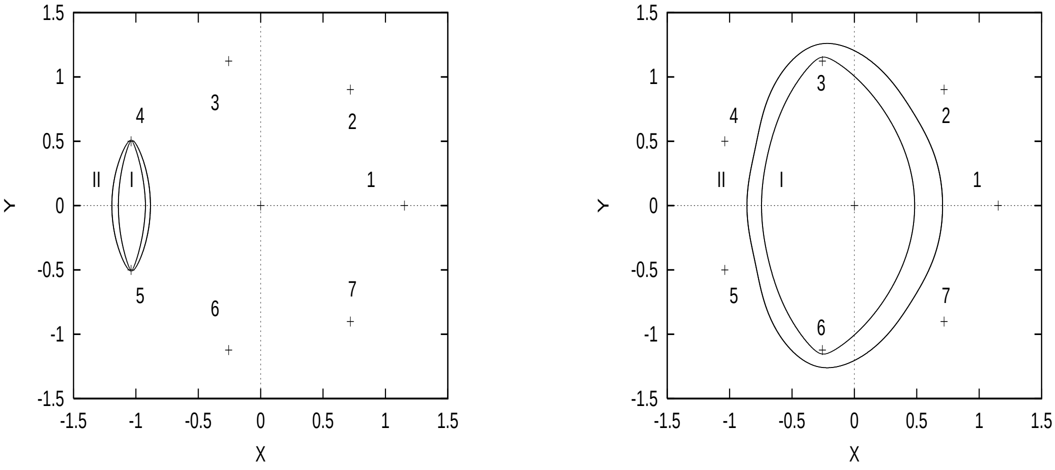

Figure 5

We conclude that the orbits of all these families, in general, are very unstable and for certain values of C the orbits collide with a peripheral.

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