Stability types of periodic orbits of multidimensional Hamiltonian systems

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Abstract

Using a suitable terminology for the different stability types of periodic orbits, we classify all the direct transitions between different stability types in Hamiltonian systems with many degrees of freedom. We also provide an indicator of how probable these transitions are.

1 Stability types of periodic orbits

Finding the periodic orbits of a dynamical system and their stability is a fundamental procedure in studying the behavior of the system. The stability or instability of a periodic orbit defines the dynamical behavior of nearby orbits. In particular non–periodic orbits near a stable periodic orbit have a time evolution similar to the one exhibited by the periodic orbit, and so their behavior is said to be ordered, while in the neighborhood of an unstable periodic orbit the system exhibit chaotic behavior.

The periodic orbits and their stability have been extensively studied in the last decades for autonomous Hamiltonian systems with 2 or 3 degrees of freedom (e.g. Hénon 1965, Broucke 1969). Considerably less work has been done for systems with more than three degrees of freedom (Howard & MacKay 1987, Howard & Dullin 1998). In the present paper we present some resent results on the stability of periodic orbits in multidimensional systems (Skokos 2001).

The linear stability or instability of a periodic orbit is defined by the eigenvalues of the corresponding monodromy matrix (see for example Yakubovich & Starzhinskii 1975). This is a matrix whose columns are suitably chosen linearly independent solutions of the so–called variational equations. These equations describe the time evolution of a small deviation from the periodic orbit.

Let us consider a periodic orbit of a $N+1$ degrees of freedom Hamiltonian system and let $\mathbf{L}$ be its monodromy matrix. We note that this Hamiltonian system corresponds to a $2N$ symplectic map in the sense that its Poincaré surface of section is a $2N$–dimensional space. The eigenvalues of $\mathbf{L}$ are roots of the characteristic polynomial $P(\lambda)$, which is a palindrome of the form (Howard & MacKay 1987):

$$P(\lambda) = \lambda^{2N} - A_{N-1}\lambda^{2N-1} + A_{N-2}\lambda^{2N-2} + ... + (-1)^N A_0 \lambda^N + ... - A_{N-1}\lambda + 1 \ . \quad (1)$$

The coefficients of P are functions of the elements of matrix $\mathbf{L}$. P is written in a simpler form in terms of the stability index

$$b = \frac{1}{\lambda} + \lambda \ . \quad (2)$$

In particular it becomes

$$Q(b) = A_0^{'} b^N - A_1^{'} b^{N-1} + ... + (-1)^{N-1} A_{N-1}^{'} b + (-1)^N A_N^{'} \ . \quad (3)$$

The polynomial $Q(b)$ is called the reduced characteristic polynomial. One of the main advantages of introducing the stability indices b_i, $i = 1, 2, ..., N$ is that they solve a polynomial of half the original order, i.e. a polynomial equation of order N. This turns the computational problem into a much more tractable one. The coefficients $A_i^{'}$, $i = 0, 1, 2, ..., N$ of $Q(b)$ are related to the roots b_i, $i = 1, 2, ..., N$. In particular $A_i^{'}$ is the sum of all possible i-tuples of b_1, ..., b_n.

The configuration of the eigenvalues of $\mathbf{L}$ on the complex plane, or equivalently the values of the stability indices determine the stability type of a periodic orbit. All the different cases are shown in Fig. 1. The orbit is stable (S) when $b \in (-2, 2)$, which means that λ and $1/\lambda$ are complex conjugate numbers on the unit circle. The orbit is unstable (U) when $b \in (-\infty, -2) \cup (2, \infty)$, which means that λ and $1/\lambda$ are real. We remark that the cases $b > 2$ and $b < -2$ are equivalent regarding the stability character of the periodic orbit, but not completely identical since a positive b cannot become negative under a continuous change of a parameter of the system. The orbit is complex unstable (Δ) when $b \in$ C - R, which means that we have four complex eigenvalues not laying on the unit circle, forming two pairs of inverse numbers and two pairs of complex conjugate numbers.

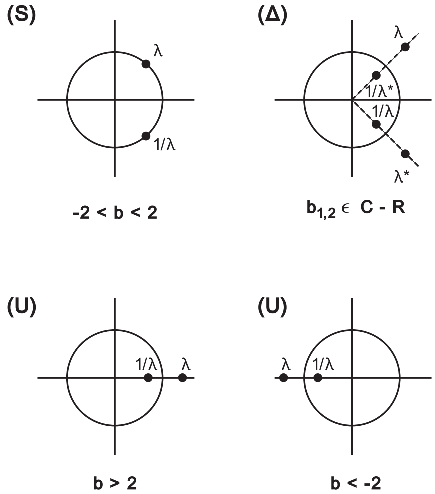

Figure 1. Configuration of the eigenvalues of matrix $\mathbf{L}$ on the complex plane, with respect to the unit circle, for the stable (S), unstable (U) and complex unstable (Δ) cases. In every case b is the corresponding stability index. We remark that λ^* denotes the complex conjugate of λ.

The general stability type of a periodic orbit of a Hamiltonian system with $N+1$ degrees of freedom, or of a $2N$–dimensional symplectic map is

$$S_n U_m \Delta_l \tag{4}$$

with n, m and l integer numbers, denoting that $2n$ eigenvalues are on the unit circle, $2m$ eigenvalues are on the real axis and $4l$ eigenvalues are on the complex plane but not on the unit circle and the real axis. In order to distinguish between the different arrangements of the eigenvalues on the real axis we can use the notation $S_n U_{m_1,m_2} \Delta_l$ with $m = m_1 + m_2$, denoting the case of having $2m_1$ negative real eigenvalues and $2m_2$ positive real eigenvalues. The integers n, m, l satisfy the inequalities:

$$0 \leq n \leq N \ , \ 0 \leq m \leq N \ , \ 0 \leq l \leq \left[\frac{N}{2} \right] \ , \tag{5}$$

and the constraint

$$n + m + 2l = N \ . \tag{6}$$

2 Direct transitions between different stability types

As already mentioned the stability of a periodic orbit is determined by the eigenvalues of the monodromy matrix $\mathbf{L}$, which depend on the coefficients A_j, $j =$

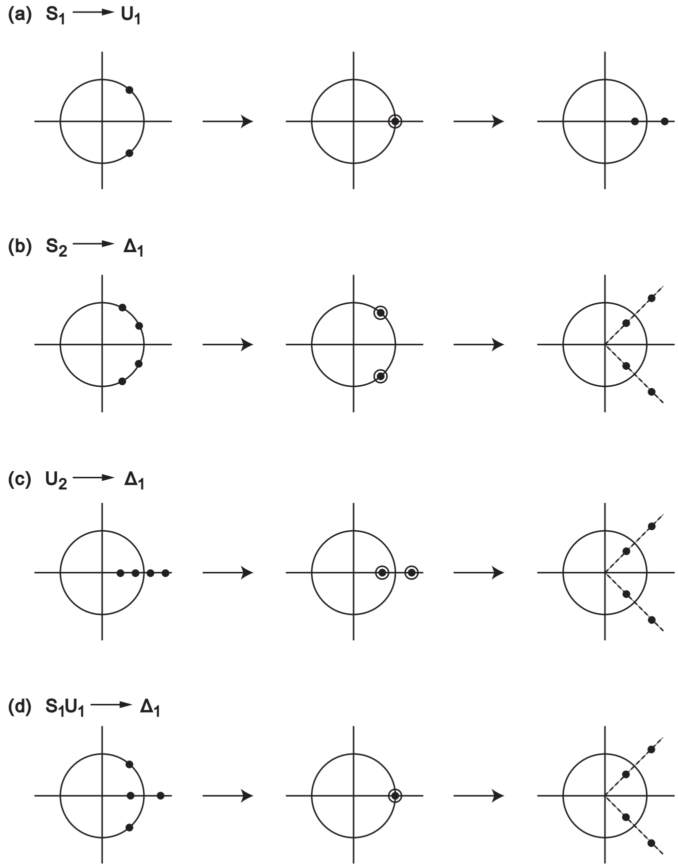

Figure 2. Schematic representations of the configuration of the eigenvalues on the complex plane for the basic transitions between different stability types, where one or two pairs of eigenvalues are involved.

$0, 1, ..., N-1$ of the characteristic polynomial $P(\lambda)$. So, the stability type of a periodic orbit is represented by a point A in the N–dimensional parameter space $\mathcal{S}$ whose coordinates are the coefficients $A_0, A_1, \dots, A_{N-1}$. As a parameter of the Hamiltonian system changes the coefficients of the characteristic polynomial also change, causing possible changes in the stability type of the periodic orbit and the motion of point A in $\mathcal{S}$.

In Fig. 2 the basic transitions between different stability types are shown schematically. These transitions happen when certain constraints on the values of the stability indices, are valid. These constraints define a transition hypersurface in the parameter space $\mathcal{S}$, the crossing of which, by point A, corresponds to the change of the stability type of the orbit. So, the transition $S_1 \to U_1$ (Fig. 2(a)) happens when b passes through $b = 2$, which corresponds to A crossing a $(N-1)$–dimensional hypersurface in $\mathcal{S}$. In a similar way $S_2 \to \Delta_1$ (Fig. 2(b)) and $U_2 \to \Delta_1$ (Fig. 2(c)) happen when point A crosses the $(N-1)$–dimensional hypersurface produced by $b_1 = b_2$, while the transition $S_1U_1 \to \Delta_1$ (Fig. 2(d)) happens when A crosses the $(N-2)$–dimensional hypersurface produced by $b_1 = b_2 = 2$. The dimension D of the hypersurface which corresponds to a certain transition is an indicator of how probable this transition is, or in other words how specific the parameters that influence the stability of an orbit must be in order for this transition to happen.

We consider now the problem of finding all the possible direct transitions between different stability types and the dimension D of the corresponding transition hypersurface in $\mathcal{S}$, without taking into account the different arrangements in the U_m case. So, we find if and how a transition of the form

$$S_n U_m \Delta_l \to S_{n+\delta n} U_{m+\delta m} \Delta_{l+\delta l} \ , \tag{7}$$

where δn, δm, δl, are the changes in the multiplicity of S, U and Δ, can happen, in the sense that there exist at least one configuration of the eigenvalues, compatible with the U_m and $U_{m+\delta m}$ types, which allows this transition. Since both the initial $(S_n U_m \Delta_l)$ and the final $(S_{n+\delta n} U_{m+\delta m} \Delta_{l+\delta l})$ stability types satisfy the constraint (6) we conclude that δn, δm, δl satisfy

$$\delta n + \delta m + 2\delta l = 0 \ . \tag{8}$$

So, the constraints (5), (6) and (8) define all the possible direct transitions of the form (7) of a Hamiltonian system with $N+1$ degrees of freedom.

Based on the simple transitions shown in Fig. 2 we find the constraints on the stability indices that define the corresponding transition hypersurface in the parameter space $\mathcal{S}$ for the general transition (7). When $\delta n \cdot \delta m \leq 0$, the multiplicity of S or U does not change or if both of them change the changes have different signs. This transition introduces $|\delta l|$ constraints of the form $b_1 = b_2$ and $|\delta l + \delta n| - |\delta l|$ constraints of the form $b = +2$ (or -2). All these constraints are independent to each other since they refer to different stability indices. So the dimension D of the corresponding transition hypersurface is

$$D = N - |\delta l + \delta n| \ . \tag{9}$$

When $\delta n \cdot \delta m > 0$, the multiplicity of both S and U increase (or decrease) leading to decrement (or increment) of the multiplicity of Δ. This means that eigenvalues come from (go to) the complex plane in quadruples. We have the following cases: (a) If δn and δm are even, the transition happens as seen in cases (b) and (c) of Fig. 2. So $|\delta l|$ constraints of the form $b_1 = b_2$ are introduced and the dimension D of the corresponding transition hypersurface is

$$D = N - |\delta l| \ . \tag{10}$$

(b) If δn and δm are odd then at least one transition of the form shown in Fig. 2(d) is needed. Thus we get $|\delta l| - 1$ constraints of the form $b_1 = b_2$ and 1 constraint of the form $b_1 = b_2 = +2$ (or -2). The dimension D of the transition hypersurface is

$$D = N - |\delta l| - 1 \ . \tag{11}$$

The dimension D of the transition hypersurface given by Eqs. (9), (10) and (11) is the maximal possible in the sense that any particular arrangement of the eigenvalues of U_m and $U_{m+\delta m}$ which is compatible with the transition (7) is performed (if it happens at all) by the crossing of a M–dimensional hypersurface with $M \leq D$.

Acknowledgments

This paper is dedicated to the memory of C. Polymilis who suggested the subject of the present paper, but unfortunately passed away before this work was completed. This work was supported by the European Union in the framework of ΕΠΕΤ II and ΚΠΣ 1994-1999, by the Research Committee of the Academy of Athens and by the Association EURATOM–Hellenic Republic under the contract ERB 5005 CT 99 0100.

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