Systematic Construction of Exact 2-D MHD Equilibria with Steady, Compressible Flow

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Abstract

We present a systematic method for constructing two-dimensional magnetohydrodynamic equilibria with compressible flow in Cartesian geometry. Using the method we find several new classes of solutions, some of which generalise known solutions, including the Kippenhahn & Schlüter (1957) and Hood & Anzer (1990) solar prominence models and the Tsinganos, Surlantzis & Priest (1993) coronal loop model with flow, and some of which are completely new. Having summarised the several classes of solutions, we give an example to illustrate the general construction method. From one of the new classes of solutions we calculate a loop-like solution, the first exact two-dimensional magnetohydrodynamic equilibrium with trans-Alfvénic flow.

1 Introduction

The investigation of plasma equilibria is one of the most important problems in magnetohydrodynamics (MHD) and arises in a number of fields including thermonuclear fusion, astrophysics and solar physics. At present there are many difficulties surrounding the description of fully three-dimensional configurations and so it is necessary to consider configurations with additional symmetry from a mathematical point of view. Even with the steady-flow and symmetry assumptions, the prospect of solving the full MHD equations is not good without further simplification. In this work we impose on our solutions spatial self-similarity i.e. the assumption of a scaling law of one of the variables on one of the coordinates. With this assumption the full MHD equations can be reduced to a system of ordinary differential equations (ODE) which can be integrated by standard methods.

In this paper we present several new classes of two-dimensional MHD equilibria in a uniform gravitational field, with flow in Cartesian geometry. The flow may be strictly sub-Alfvénic, super-Alfvénic or trans-Alfvénic. In the trans-Alfvénic case the solution can be made to cross the Alfvén critical point smoothly. The solutions are calculated using a systematic nonlinear separation of variables construction method already seen in spherical geometry (Vlahakis & Tsinganos, 1998). The resulting classes of solutions are found to include as special cases several well-known two-dimensional equilibria as well as many hitherto unknown solutions. To illustrate the general construction method we present an example solution of a loop-like structure with unidirectional flow.

The paper is organised as follows. We give details of our assumptions in Section 2 before using a simple theorem to construct several classes of translationally self-similar solutions in Section 3, summarised in Table 1. In Section 4 a special case of one of the classes is given. The paper is summarised in Section 5.

2 Assumptions

In order to construct analytically some classes of exact solutions for coronal loops, we shall first assume that the magnetic loop is planar, $B_y = 0$, and there exists only a velocity component along the y-direction which is constant on each poloidal field line, $V_y = \Phi_A(A) = L(A)$. These assumptions are necessary for the solution to be finite at the Alfvén point $M^2 = 1$. To proceed further we shall make the following two key assumptions:

1. that the Alfvén number M is solely a function of the dimensionless horizontal distance $x = X/Z_0$, i.e.,

$$M^2 = \frac{4\pi\rho V_p^2}{B_p^2} = M^2(x)\,, \tag{1}$$

and

2. that the poloidal velocity and magnetic fields have an exponential dependence on $z = Z/Z_0$,

$$A = Z_0 B_0 \mathcal{A}(\alpha)\ , \qquad \alpha = G(x)\exp(-z)\,, \tag{2}$$

for some function $G(x)$, where Z_0 and B_0 are constants.

With these assumptions the two-dimensional MHD equations can be reduced to a single equation of the form

$$Y_n(\alpha)X_n(x) + \ldots + Y_1(\alpha)X_1(x) = 0\,. \tag{3}$$

3 Construction of models

Eq.(3) can be solved with the help of the following theorem (Vlahakis 1998).
Theorem: Let $F_n(\alpha)$, $Y_i(\alpha)$ and $X_i(x)$, $i = 1, 2, \ldots, n$ be functions of α and x such that the relation

$$F_n(\alpha) = Y_1(\alpha)X_1(x) + Y_2(\alpha)X_2(x) + \ldots + Y_n(\alpha)X_n(x) \tag{4}$$

holds. Then there exist constants $c_1, c_2, \ldots, c_n$ such that

$$F_n(\alpha) = c_1 Y_1(\alpha) + c_2 Y_2(\alpha) + \ldots + c_n Y_n(\alpha). \tag{5}$$

Either (i) $X_n(x) = 0$ for all x in which case (indicated by the digit "0") we have

$$Y_{n-1}(\alpha)X_{n-1}(x) + \ldots + Y_1(\alpha)X_1(x) = 0\,, \tag{6}$$

or (ii) $X_n(x) \neq 0$ in which case (indicated by the digit "1") we have

$$Y_n(\alpha) = -\frac{X_1(x)}{X_n(x)}Y_1(\alpha) - \ldots - \frac{X_{n-1}(x)}{X_n(x)}Y_{n-1}(\alpha)\,. \tag{7}$$

Therefore according to the Theorem there exist constants $\mu_i^{(n)}$, $i = 1, \ldots, n-1$ such that $Y_n(\alpha) = \sum_{i=1}^{n-1} \mu_i^{(n)} Y_i(\alpha)$ giving a condition between the functions of α.

In both cases (i) and (ii) we obtain a sum with $n-1$ terms and we can continue to apply this algorithm until we have only one term. Since for each of the n terms in Eq.(3) there are the above two possibilities we may obtain 2^n cases each corresponding to a number $xx\ldots xx$ where each x is either 0 or 1. The number of 1's is the number of conditions between functions of α while the number of 0's is the number of conditions between functions of x.

Table 1: Translationally self-similar exact MHD solutions

	α − integrals	ODE	pressure
1.	$g_1 = C_1\alpha^2 + C_2\alpha^\lambda$ $g_2 = D_1\alpha^2 + D_2\alpha^\lambda$	$C_1 f_2' - 2C_1 f_4 + D_1 f_3' - 2D_1 f_5 = 0$ $C_2 f_2' - \lambda C_2 f_4 + D_2 f_3' - \lambda D_2 f_5 = 0$ $f_0' = 0$	$\frac{4\pi}{B_0^2} P = P_0 + P_1\alpha^2 + P_2\alpha^\lambda$ $P_0 = f_0$ $P_1 = C_1 f_2 + D_1 f_3 - 2C_1 f_1$ $P_2 = C_2 f_2 + D_2 f_3 - \lambda C_2 f_1$
2.	$g_1 = C_1\alpha^2 + C_2 \ln\alpha$ $g_2 = D_1\alpha^2 + D_2 \ln\alpha$	$C_1 f_2' - 2C_1 f_4 + D_1 f_3' - 2D_1 f_5 = 0$ $C_2 f_2' + D_2 f_3' = 0$ $f_0' - C_2 f_4 - D_2 f_5 = 0$	$\frac{4\pi}{B_0^2} P = P_0 + P_1\alpha^2 + P_2 \ln\alpha$ $P_0 = f_0 - C_2 f_1$ $P_1 = C_1 f_2 + D_1 f_3 - 2C_1 f_1$ $P_2 = C_2 f_2 + D_2 f_3$
3.	$g_1 = C_1 \ln\alpha + C_2 (\ln\alpha)^2$ $g_2 = D_1 \ln\alpha + D_2 (\ln\alpha)^2$	$C_1 f_2' - 2C_2 f_4 + D_1 f_3' - 2D_2 f_5 = 0$ $C_2 f_2' + D_2 f_3' = 0$ $f_0' - C_1 f_4 - D_1 f_5 = 0$	$\frac{4\pi}{B_0^2} P = P_0 + P_1 \ln\alpha + P_2 (\ln\alpha)^2$ $P_0 = f_0 - C_1 f_1$ $P_1 = C_1 f_2 + D_1 f_3 - 2C_2 f_1$ $P_2 = C_2 f_2 + D_2 f_3$
4.	$g_1 = C_1\alpha^2 + C_2\alpha^2 \ln\alpha$ $g_2 = D_1\alpha^2 + D_2\alpha^2 \ln\alpha$	$C_1 f_2' - (2C_1 + C_2) f_4 + D_1 f_3' - (2D_1 + D_2) f_5 = 0$ $C_2 f_2' - 2C_2 f_4 + D_2 f_3' - 2D_2 f_5 = 0$ $f_0' = 0$	$\frac{4\pi}{B_0^2} P = P_0 + P_1\alpha^2 + P_2\alpha^2 \ln\alpha$ $P_0 = f_0$ $P_1 = C_1 f_2 + D_1 f_3 - (2C_1 + C_2) f_1$ $P_2 = C_2 f_2 + D_2 f_3 - 2C_2 f_1$

From our relation (3) we can obtain 2^5 solutions, each corresponding to a number $xxxxx$. However many of them are not of interest to us at present.

Our final results for the free integrals and the corresponding ODEs with respect to x are summarised in Table 1. The expressions for the pressure in each case are also given. Some families include as special cases the Kippenhahn & Schlüter (1957) and Hood & Anzer (1990) solar prominence models and the Tsinganos, Surlantzis & Priest (1993) coronal loop model with flow (see Petrie et al., 2001).

4 A new class of solutions

It seems that a particularly interesting new case is the first family in Table 1. In this section we give an example of this case, with $C_2 = 0$.

In the sub-Alfvénic case the equations have no critical points and can be integrated without difficulty. In the trans-Alfvénic case, however, the equations have critical points where $M^2 = 1$. Loop-like solutions of both catagories are possible (see Petrie et al., 2001). Here we give a trans-Alfvénic example. In order to obtain a regular trans-Alfvénic solution it is necessary to fix the parameters so that the ODEs have no singularity at $M^2 = 1$, but the parameters cannot be fixed independently of the solution. The correct combination of parameter values can be found by a shooting method as seen in Vlahakis & Tsinganos (1999).

Example field lines and plasma parameter profiles the example are shown in Fig. 1. The horizontal and vertical coordinates are given in units of Z_0. The field lines are all loop-like in shape and the translational self-similarity of the solutions causes any two field lines to differ from each other only by a constant value in the z-direction. In Fig. 1, all quantities can be seen to have symmetric profiles. The pressure, density and magnetic field strength along the field line all have their minima at the apex and increase monotonically towards the foot points since these quantities all decrease exponentially with height. Note that the magnetic pressure is greater than the plasma pressure, that the velocity is clearly trans-Alfvénic, being super-Alfvénic near the apex and sub-Alfvénic near the footpoints, and that the critical points are crossed smoothly.

5 Conclusions

We have presented a systematic method for constructing exact two-dimensional MHD equilibria with compressible flow in Cartesian geometry with a view to modelling structures in the solar atmosphere. Using this method we have found several new classes of solutions, some of which include some well-known exact loop and prominence models as special cases and some of which are completely new. To illustrate the method we have given a simple loop-like solution. The critical points of this trans-Alfvénic example are crossed smoothly and all quantities in the model are regular. Some solutions have physical properties not previously available in

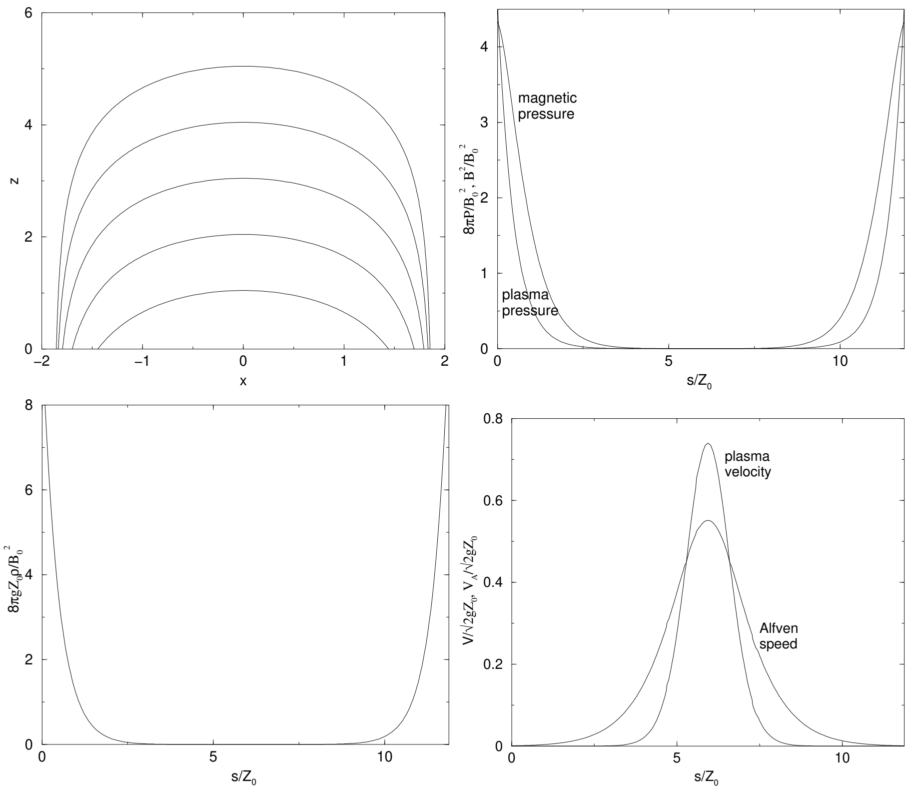

Figure 1: Plots of field lines (top left) and plasma quantities for illustrative example: plasma and magnetic pressures (top right), plasma density (bottom left) and plasma velocity and Alfvén speed (bottom right) along the largest loop shown in the field lines plot (top left). See text for discussion.

exact two-dimensional MHD work. It seems likely that the solution construction method may be applied in future in providing quantitative models of prominences and arcades to be compared with observations or giving details of the heating profile of the structure. Such applications will be the subject of future work.

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References

Hood A.W., Anzer U., 1990, Solar Phys. 126, 117
Kippenhahn R., Schlüter A., 1957, Z. Astrophys. 43, 36
Petrie G.J.D., Vlahakis V., Tsinganos K., 2001, A&A in press
Tsinganos K., Surlantzis G., Priest E.R., 1993, A&A 275, 613
Vlahakis N. 1998, Analytical Modeling of Cosmic Winds and Jets, PhD Thesis, University of Crete
Vlahakis N., Tsinganos K., 1998, MNRAS 298, 777
Vlahakis N., Tsinganos K., 1999, MNRAS 307, 279